

On the Well-posedness of the Problem of Reconstruction of Non-separate Boundary Conditions

Akhymov A. M.^a, Mouftakhov A. V.^b, Teicher M.^b, Yamilova L. S.^c

^a Institute of mechanics, Ufa, Russia;

^b Department of Mathematics, Bar-Ilan University, Ramat-Gan, Israel;

^c Department of Differential Equations, Bashkir State University, Ufa, Russia.

1 Introduction

The identification of boundary conditions of a spectral problem is an important practical problem.

The question arises whether one would be able to detect boundary conditions, using finite number eigenvalues. The following papers give and substantiate a positive answer to this question for several cases (see [1], [2], [3], [4], [5]). In this paper we continue these researches.

The problem in question belongs to the class of inverse problems and is a completely natural problem of identification of the boundary conditions.

The problem of determining a boundary condition has been considered in [7]. However, as data for finding the boundary conditions and as opposed to condensation and inversion (as in [7]), we take a set of eigenvalues.

Similarly formulated problems also occur in the spectral theory of differential operators, where it is required to determine the coefficients of a differential equation and the boundary conditions using a set of eigenvalues (for more details, see [6], [10], [11], [13], [14], [15]). However, as data for finding the boundary conditions, we take one spectrum, not several spectra or other additional spectral data (for example, the spectral function, the Weyl function or the so-called weighting numbers), that were used in these papers. Moreover, their principal aim was to determine the coefficients in the equation, not in boundary conditions. The aim of this work is to determine the boundary conditions of the eigenvalue problem, from its spectrum in the case of a known differential equation.

We consider an inverse spectral problem with the third-order differential equation and the non-separated boundary conditions.

Two theorems on the uniqueness of the solution of this problem are proved, and a method for establishing the unknown conditions is obtained, using 19 eigenvalues.

The method of approximate calculation of unknown boundary conditions is explained, with the help of an example.

2 Formulation of the Inverse Problem

The following spectral problem is considered:

$$l(y) = y'''(x) + \lambda p_1 y'(x) + \lambda^2 p_2 y'(x) + \lambda^3 p_3 y(x) = 0, \quad (1)$$

$$U_i(y) = \sum_{k=1}^3 \left(a_{ik} y^{(k-1)}(0) + a_{i\ k+3} y^{(k-1)}(1) \right) = 0 \quad (i = 1, 2, 3), \quad (2)$$

where a_{ik} and p_i , are not dependent on parameter λ , and $a_{ik}, p_i \in \mathbb{C}$.

We denote the matrix formed by the coefficients a_{ij} , by A , and its third-order minors, by M_{ijk} :

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{16} \\ a_{21} & a_{22} & a_{23} & \dots & a_{26} \\ a_{31} & a_{32} & a_{33} & \dots & a_{36} \end{vmatrix}, \quad M_{ijk} = \begin{vmatrix} a_{1i} & a_{1j} & a_{1k} \\ a_{2i} & a_{2j} & a_{2k} \\ a_{3i} & a_{3j} & a_{3k} \end{vmatrix}$$

$$(1 \leq i < j < k \leq 6).$$

In terms of problem (1)–(2), the inverse problem of reconstructing the boundary conditions (2) can be stated as follows:

Inverse problem. The coefficients a_{ij} of the forms $U_i(y)$, ($i = 1, 2, 3$) in problem (1)–(2), are unknown. The rank of the matrix A formed by these coefficients is equal to 3. The non-zero eigenvalues λ_k of problem (1)–(2), are known. It is required to find the boundary condition, i.e. to reconstruct the linear span $\langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \rangle$ of vectors $\mathbf{a}_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4}, a_{i5}, a_{i6})$, ($i = 1, 2, 3$).

3 Uniqueness Theorem for the Solution of the Inverse Problem

Along with the problem (1)–(2), consider the following spectral problem

$$l(y) = y'''(x) + \lambda p_1 y'(x) + \lambda^2 p_2 y'(x) + \lambda^3 p_3 y(x) = 0, \quad (3)$$

$$\tilde{U}_i(y) = \sum_{k=1}^3 \left(b_{ik} y^{(k-1)}(0) + b_{i\ k+3} y^{(k-1)}(1) \right) = 0 \quad (i = 1, 2, 3). \quad (4)$$

We denote the matrix formed by the coefficients b_{ij} , by B , and its third-order minors, by \widetilde{M}_{ijk} :

$$B = \begin{vmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{16} \\ b_{21} & b_{22} & b_{23} & \dots & b_{26} \\ b_{31} & b_{32} & b_{33} & \dots & b_{36} \end{vmatrix}, \quad \widetilde{M}_{ijk} = \begin{vmatrix} b_{1i} & b_{1j} & b_{1k} \\ b_{2i} & b_{2j} & b_{2k} \\ b_{3i} & b_{3j} & b_{3k} \end{vmatrix}.$$

Theorem 1 (on uniqueness of the solution of the inverse problem). *Suppose that the following conditions are satisfied:*

rank $A = \text{rank } B = 3$, not every number λ , is an eigenvalue for problems (1)–(2), (3)–(4), the eigenvalues of the problem (1)–(2), and the eigenvalues of the problem (3)–(4), coincide with their multiplicities taken into account, and roots of the characteristic equation $\omega^3 + p_1 \omega^2 + p_2 \omega + p_3 = 0$, satisfy conditions:

- 1) $\sum_{i \in I_k} e_i \omega_i \neq 0$, where $e_i = \pm 1$ and I_k is any subset of the set $\{1, 2, 3\}$,
- 2) $p_2 \neq 0$.

Then $\text{Span}\langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \rangle = \text{Span}\langle \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \rangle$, where $\mathbf{a}_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4}, a_{i5}, a_{i6})$, $\mathbf{b}_i = (b_{i1}, b_{i2}, b_{i3}, b_{i4}, b_{i5}, b_{i6})$, ($i = 1, 2, 3$).

Remark. *It follows from the theorem's conditions that $p_1 \neq 0$, $p_3 \neq 0$.*

Proof.

By definition, put $z_j = \begin{cases} y^{(j-1)}(0), & j = 1, 2, 3; \\ y^{(j-4)}(1), & j = 4, 5, 6. \end{cases}$

Then problems (1)–(2), (3)–(4), can be represented as

$$l(y) = 0, \quad U_i(y) = \sum_{j=1}^6 a_{ij} z_j = 0 \quad (i = 1, 2, 3). \quad (5)$$

$$l(y) = 0, \quad \widetilde{U}_i(y) = \sum_{j=1}^6 \widetilde{a}_{ij} z_j = 0 \quad (i = 1, 2, 3). \quad (6)$$

Let $\{y_k(x)\}_{k=1,2,3}$, be a fundamental system of the solution of equation (1), which meet the following conditions:

$$y_k^{j-1}(0) = \begin{cases} 1, & k = j, \\ 0, & k \neq j \end{cases} \quad (k, j = 1, 2, 3).$$

$$\text{By definition, put } z_{kj} = \begin{cases} y_k^{(j-1)}(0), & j = 1, 2, 3, \\ y_k^{(j-4)}(1), & j = 4, 5, 6 \end{cases} \quad (k = 1, 2, 3).$$

By $\Delta(\lambda)$, denote the characteristic determinant of problem (5). By $\widetilde{\Delta}(\lambda)$, denote the characteristic determinant of problem (6).

Let's consider the following boundary problem

$$l(y) = 0, \quad \widetilde{U}_i(y) = \sum_{j=1}^6 b_{ij} z_j = 0 \quad (i = 1, 2), \quad (7)$$

$$e^{f(\lambda)} \tilde{U}_3(y) = e^{f(\lambda)} \sum_{j=1}^6 b_{3j} z_j, \quad (8)$$

where $f(\lambda)$ is an entire function, which will be chosen later.

The following function is a characteristic determinant of the problem (7)–(8).

$$\tilde{\Delta}_1(\lambda) = \begin{vmatrix} \tilde{U}_1(z_1) & \tilde{U}_1(z_2) & \tilde{U}_1(z_3) \\ \tilde{U}_2(z_1) & \tilde{U}_2(z_2) & \tilde{U}_2(z_3) \\ e^{f(\lambda)} \tilde{U}_3(z_1) & e^{f(\lambda)} \tilde{U}_3(z_2) & e^{f(\lambda)} \tilde{U}_3(z_3) \end{vmatrix} \equiv e^{f(\lambda)} \tilde{\Delta}(\lambda). \quad (9)$$

It immediately follows that the eigenvalues of spectral problems (6) and (7) – (8), are equal.

Hence, by the condition of the theorem, the eigenvalues of spectral problems (5) and (7) – (8), are equal as well.

$\Delta(\lambda)$ and $\tilde{\Delta}_1(\lambda)$ are entire functions of λ (see [12]). The eigenvalues of spectral problems (5), are zeros of function $\Delta(\lambda)$, and the eigenvalues of spectral problems (7) – (8) are zeros of function $\tilde{\Delta}_1(\lambda)$. Since the eigenvalues of spectral problems (5) and (7) – (8) are equal, zeros of functions $\Delta(\lambda)$ and $\tilde{\Delta}_1(\lambda)$, are equal too. Hence, by consequence of Weierstrass' factorization theorem, about zeros of an entire function [9],

$$\tilde{\Delta}_1(\lambda) \equiv C e^{g(\lambda)} \Delta(\lambda), \quad (10)$$

where $g(\lambda)$ is an entire function.

It follows from (10) that

$$e^{f(\lambda)} \begin{vmatrix} U_1(y_1) & U_1(y_2) & U_1(y_3) \\ U_2(y_1) & U_2(y_2) & U_2(y_3) \\ U_3(y_1) & U_3(y_2) & U_3(y_3) \end{vmatrix} \equiv C e^{g(\lambda)} \begin{vmatrix} \tilde{U}_1(y_1) & \tilde{U}_1(y_2) & \tilde{U}_1(y_3) \\ \tilde{U}_2(y_1) & \tilde{U}_2(y_2) & \tilde{U}_2(y_3) \\ \tilde{U}_3(y_1) & \tilde{U}_3(y_2) & \tilde{U}_3(y_3) \end{vmatrix}. \quad (11)$$

$f(\lambda)$ is an entire function. Let's choose $f(\lambda)$, so that $f(\lambda) \equiv g(\lambda)$. Then it follows from (11) that

$$\begin{vmatrix} U_1(y_1) & U_1(y_2) & U_1(y_3) \\ U_2(y_1) & U_2(y_2) & U_2(y_3) \\ U_3(y_1) & U_3(y_2) & U_3(y_3) \end{vmatrix} \equiv C \begin{vmatrix} \tilde{U}_1(y_1) & \tilde{U}_1(y_2) & \tilde{U}_1(y_3) \\ \tilde{U}_2(y_1) & \tilde{U}_2(y_2) & \tilde{U}_2(y_3) \\ \tilde{U}_3(y_1) & \tilde{U}_3(y_2) & \tilde{U}_3(y_3) \end{vmatrix}. \quad (12)$$

Taking into consideration the boundary conditions of the problems (5) and (6), It follows from (12) that

$$\sum_{i=1}^6 \sum_{j=1}^6 \sum_{k=1}^6 z_{1i} z_{2j} z_{3k} M_{ijk} = C \sum_{i=1}^6 \sum_{j=1}^6 \sum_{k=1}^6 z_{1i} z_{2j} z_{3k} \tilde{M}_{ijk}. \quad (13)$$

It follows from (13) that

$$\sum_{1 \leq i < j < k \leq 6} \left(M_{ijk} - C \widetilde{M}_{ijk} \right) Z_{ijk} = 0, \quad (14)$$

where $Z_{ijk} = \begin{vmatrix} z_{1i} & z_{1j} & z_{1k} \\ z_{2i} & z_{2j} & z_{2k} \\ z_{3i} & z_{3j} & z_{3k} \end{vmatrix}.$

$\{Z_{ijk} | 1 \leq i < j < k \leq 6\}$ is a system of linearly independent functions of λ . Then it follows from (14) that $(M_{ijk} - C \widetilde{M}_{ijk}) = 0$, i.e. $M_{ijk} = C \widetilde{M}_{ijk}$. Then $\text{Span}\langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \rangle = \text{Span}\langle \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \rangle$ (see [8]). This completes the proof of the theorem.

4 Exact Solution of the Inverse Problem

Let us assume that λ_k , ($k = 1, \dots, 19$), are eigenvalues of spectral problems (1)–(2)

$$y(x, \lambda) = C_1 y_1(x, \lambda) + C_2 y_2(x, \lambda) + C_3 y_3(x, \lambda), \quad (15)$$

where $\{y_i(x) | i = 1, 2, 3\}$ is a fundamental system of the solution of equation (1)–(2).

We shall find C_i , ($i = 1, 2, 3$), using the boundary condition (2). We substitute (15) with (2). From this, we obtain the following system of equations:

$$\begin{aligned} C_1 U_1(y_1) + C_2 U_1(y_2) + C_3 U_1(y_3) &= 0, \\ C_1 U_2(y_1) + C_2 U_2(y_2) + C_3 U_2(y_3) &= 0, \\ C_1 U_3(y_1) + C_2 U_3(y_2) + C_3 U_3(y_3) &= 0. \end{aligned}$$

The non-zero solution for C_i exists if and only if the following determinant

$$\Delta(\lambda) = \begin{vmatrix} U_1(y_1) & U_1(y_2) & U_1(y_3) \\ U_2(y_1) & U_2(y_2) & U_2(y_3) \\ U_3(y_1) & U_3(y_2) & U_3(y_3) \end{vmatrix} \quad (16)$$

is equal to zero [12].

Transforming (16) and using the designations entered at the proof of Theorem 1, we obtain

$$\Delta(\lambda) = \sum_{1 \leq i < j < k \leq 6} Z_{ijk}(\lambda) M_{ijk} = 0, \text{ where } Z_{ijk} = \begin{vmatrix} z_{1i} & z_{1j} & z_{1k} \\ z_{2i} & z_{2j} & z_{2k} \\ z_{3i} & z_{3j} & z_{3k} \end{vmatrix}. \quad (17)$$

Let's substitute values λ_m ($m = 1, \dots, 19$) to $\Delta(\lambda)$, we shall receive the system with 19 homogeneous equations of 20 variables M_{ijk} .

$$\sum_{1 \leq i < j < k \leq 6} Z_{ijk}(\lambda_m) M_{ijk} = 0. \quad (18)$$

The system (18) has infinite number of solutions. If the rank of the system equals 19, then unknown minors M_{ijk} are determined from the system, accurate to coefficient. By these minors, the corresponding boundary conditions can be unequivocally found by means of known methods of linear algebra.

Indeed, since $\text{rank} A = 3$ then one of minors M_{ijk} is not equal to zero. Let $M_{135} \neq 0$, then after linear transformations the matrix A can be written as follows:

$$A = \begin{vmatrix} 1 & a_{12} & 0 & a_{14} & 0 & a_{16} \\ 0 & a_{22} & 1 & a_{24} & 0 & a_{26} \\ 0 & a_{32} & 0 & a_{34} & 1 & a_{36} \end{vmatrix}.$$

With that, the minors M_{ijk} will not exchange (or probably will be multiplied by non-zero number).

For this matrix, we get

$$M_{135} = 1, M_{134} = a_{34}, M_{136} = a_{36}, M_{123} = -a_{32}, M_{235} = a_{12},$$

$$M_{156} = -a_{26}, M_{145} = a_{24}, M_{345} = -a_{14}, M_{125} = a_{22}, M_{356} = a_{16}.$$

Then matrix A can be written as follows:

$$A = \begin{vmatrix} M_{135} & M_{235} & 0 & -M_{345} & 0 & M_{356} \\ 0 & M_{125}/M_{135} & 1 & M_{145}/M_{135} & 0 & -M_{156}/M_{135} \\ 0 & -M_{123}/M_{135} & 0 & M_{134}/M_{135} & 1 & M_{136}/M_{135} \end{vmatrix} \quad (19)$$

This reasoning proves:

Theorem 2 (on the uniqueness of the solution of the inverse problem). *If the matrix of system (18) has a rank of 19, the solution of the inverse problem of the reconstruction boundary conditions (2) is unique.*

5 Example

We shall consider application of a method of definition of the boundary conditions by 19 eigenvalues for the following boundary problem:

$$l(y) = y'''(x) - (3i + 3)\lambda y''(x) + (9i - 2)\lambda^2 y'(x) + 6\lambda^3 y(x) = 0, \quad (20)$$

$$U_i(y) = \sum_{k=1}^3 \left(a_{ik} y^{(k-1)}(0) + a_{i,k+3} y^{(k-1)}(1) \right) = 0, \quad (i = 1, 2, 3). \quad (21)$$

Let us know 19 eigenvalues of a problem (20) - (21)

$$\begin{aligned} \lambda_1 &= 0.46 - 0.12 \times i, \quad \lambda_2 = 5.88 + 3.86 \times i, \quad \lambda_3 = 6.51 - 0.55 \times i, \\ \lambda_4 &= 12.81 - 0.56 \times i, \quad \lambda_5 = 19.1 - 0.56 \times i, \quad \lambda_6 = -4.27 + 0.51 \times i, \\ \lambda_7 &= -7.16 + 1.06 \times i, \quad \lambda_8 = -10.54 + i, \quad \lambda_9 = -13.50 + 1.32 \times i, \\ \lambda_{10} &= -19.81 + 1.49 \times i, \quad \lambda_{11} = -23.1 + 1.41 \times i, \quad \lambda_{12} = -26.11 + 1.61 \times i, \\ \lambda_{13} &= -29.38 + 1.54 \times i, \quad \lambda_{14} = -32.41 + 1.71 \times i, \quad \lambda_{15} = -35.67 + 1.64 \times i, \\ \lambda_{16} &= -38.7 + 1.8 \times i, \quad \lambda_{17} = -44.99 + 1.87 \times i, \quad \lambda_{18} = -48.23 + 1.80 \times i, \\ \lambda_{19} &= -51.28 + 1.93 \times i. \end{aligned}$$

The fundamental system of decisions of the equation (20) satisfying conditions

$$y_k^{j-1}(0) = \begin{cases} 1, & k = j \\ 0, & k \neq j, \end{cases} \quad (k, j = 1, 2, 3)$$

has the following appearance:

$$\begin{aligned} y_1 &= C_1 e^{i\lambda x} + C_2 e^{2i\lambda x} + C_3 e^{3\lambda x}, \quad y_2 = K_1 e^{i\lambda x} + K_2 e^{2i\lambda x} + K_3 e^{3\lambda x}, \\ y_3 &= N_1 e^{i\lambda x} + N_2 e^{2i\lambda x} + N_3 e^{3\lambda x}, \end{aligned}$$

where

$$\begin{aligned} C_1 &= \frac{9}{5} + \frac{3}{5} i, \quad C_2 = -\frac{9}{13} - \frac{6}{13} i, \quad C_3 = -\frac{7}{65} - \frac{9}{65} i, \\ K_1 &= \left(-\frac{9}{10} + \frac{7}{10} i\right)/\lambda, \quad K_2 = \left(\frac{9}{13} - \frac{7}{13} i\right)/\lambda, \quad K_3 = \left(\frac{27}{130} - \frac{21}{130} i\right)/\lambda, \\ N_1 &= \left(\frac{1}{10} - \frac{3}{10} i\right)/\lambda^2, \quad N_2 = \left(-\frac{2}{13} + \frac{3}{13} i\right)/\lambda^2, \quad N_3 = \left(\frac{7}{130} + \frac{9}{130} i\right)/\lambda^2. \end{aligned}$$

Having solved system (18) by Maple, we shall find minors M_{ijk} :

$$\begin{aligned} M_{135} &= C, \quad M_{236} = (-5.65 - 3.95 i) \times 10^{-19} C, \\ M_{134} &= 0.50 C - 2.00 i \times 10^{-8} C, \quad M_{145} = (5.05 + 3.62 i) \times 10^{-8} C, \\ M_{136} &= (-5.04 - 3.30 i) \times 10^{-10} C, \quad M_{126} = (-7.73 + 7.10 i) \times 10^{-10} C, \\ M_{156} &= (2.44 + 0.56 i) \times 10^{-7} C, \quad M_{256} = (-0.03 - 1.21 i) \times 10^{-8} C, \\ M_{356} &= C + 1.10 i \times 10^{-8} C, \quad M_{346} = 0.5 C - 6.86 i \times 10^{-9} C, \\ M_{124} &= (1.24 - 5.13 i) \times 10^{-8} C, \quad M_{456} = (-1.84 - 0.52 i) \times 10^{-7} C, \\ M_{125} &= (-2.17 - 3.30 i) \times 10^{-8} C, \quad M_{146} = (0.85 - 3.79 i) \times 10^{-8} C, \\ M_{235} &= C + 1.13 i \times 10^{-8} C, \quad M_{123} = (1.29 + 1.33 i) \times 10^{-9} C, \\ M_{345} &= -0.5 C - 8.10 i \times 10^{-8} C, \quad M_{234} = 0.5 C - 8.20 i \times 10^{-9} C, \end{aligned}$$

$$M_{245} = (0.63 - 2.74i) \times 10^{-8}C, \quad M_{246} = (6.63 - 1.27i) \times 10^{-8}C.$$

Let $C = 1$, then we have

$$M_{125} \approx 0, \quad M_{145} \approx 0, \quad M_{156} \approx 0, \quad M_{123} \approx 0, \quad M_{136} \approx 0.$$

By (19), we get

$$A = \begin{pmatrix} 1 & 1 & 0 & 0.5 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 1 & 0 \end{pmatrix}.$$

Thus, unknown boundary conditions are found.

$$U_1(y) = y(0) + y'(0) + 0.5y(1) + y''(1) = 0,$$

$$U_2(y) = y''(0) = 0, \quad U_1(y) = 0.5y(1) + y'(1) = 0.$$

6 Acknowledgements

This research was partially supported by the Russian Foundation for Basic Research (06-01-00354a), Emmy Noether Research Institute for Mathematics, the Minerva Foundation of Germany, the Excellency Center "Group Theoretic Methods in the Study of Algebraic Varieties" of the Israel Science Foundation, and by EAGER (European Network in Algebraic Geometry).

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